

Structure theory for maximally monotone operators with points of continuity

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Abstract

In this paper, we consider the structure of maximally monotone operators in Banach space whose domains have nonempty interior and we present new and explicit structure formulas for such operators. Along the way, we provide new proofs of the norm-to-weak* closedness and of property (Q) for these operators (as recently proven by Voisei). Various applications and limiting examples are given.

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1 Introduction

We assume throughout that X is a real Banach space with norm $\|\cdot\|$, that X^* is the continuous dual of X , and that X and X^* are paired by $\langle \cdot, \cdot \rangle$. The *closed unit ball* in X is

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denoted by $B_X := \{x \in X \mid \|x\| \leq 1\}$, $B_\delta(x) := x + \delta B_X$ (where $\delta > 0$ and $x \in X$) and $\mathbb{N} = \{1, 2, 3, \dots\}$.

Let $A: X \rightrightarrows X^*$ be a *set-valued operator* (also known as a relation, point-to-set mapping or multifunction) from X to X^* , i.e., for every $x \in X$, $Ax \subseteq X^*$, and let $\text{gra } A := \{(x, x^*) \in X \times X^* \mid x^* \in Ax\}$ be the *graph* of A . The *domain* of A is $\text{dom } A := \{x \in X \mid Ax \neq \emptyset\}$ and $\text{ran } A := A(X)$ is the *range* of A .

Recall that A is *monotone* if

$$(1) \quad \langle x - y, x^* - y^* \rangle \geq 0, \quad \forall (x, x^*) \in \text{gra } A \quad \forall (y, y^*) \in \text{gra } A,$$

and *maximally monotone* if A is monotone and A has no proper monotone extension (in the sense of graph inclusion). Let $A: X \rightrightarrows X^*$ be monotone and $(x, x^*) \in X \times X^*$. We say (x, x^*) is *monotonically related to* $\text{gra } A$ if

$$\langle x - y, x^* - y^* \rangle \geq 0, \quad \forall (y, y^*) \in \text{gra } A.$$

Monotone operators have frequently proven to be a key class of objects in both modern Optimization and Analysis; see, e.g., [5, 6, 8], the books [4, 13, 15, 22, 29, 30, 25, 27, 37, 38, 39] and the references given therein.

As much as possible we adopt standard convex analysis notation. Given a subset C of X , $\text{int } C$ is the *interior* of C and \overline{C} is the *norm closure* of C . For the set $D \subseteq X^*$, \overline{D}^{w^*} is the weak* closure of D , and the norm \times weak* closure of $C \times D$ is $\overline{C \times D}^{\|\cdot\| \times w^*}$. The *indicator function* of C , written as ι_C , is defined at $x \in X$ by

$$(2) \quad \iota_C(x) := \begin{cases} 0, & \text{if } x \in C; \\ +\infty, & \text{otherwise.} \end{cases}$$

For every $x \in X$, the *normal cone* operator of C at x is defined by $N_C(x) := \{x^* \in X^* \mid \sup_{c \in C} \langle c - x, x^* \rangle \leq 0\}$, if $x \in C$; and $N_C(x) := \emptyset$, if $x \notin C$; the *tangent cone* operator of C at x is defined by $T_C(x) := \{x \in X \mid \sup_{x^* \in N_C(x)} \langle x, x^* \rangle \leq 0\}$, if $x \in C$; and $T_C(x) := \emptyset$, if $x \notin C$. The *hypertangent cone* of C at x , $H_C(x)$, coincides with the interior of $T_C(x)$ (see [12, 11]).

Let $f: X \rightarrow]-\infty, +\infty]$. Then $\text{dom } f := f^{-1}(\mathbb{R})$ is the *domain* of f . We say f is proper if $\text{dom } f \neq \emptyset$. Let f be proper. The *subdifferential* of f is defined by

$$\partial f: X \rightrightarrows X^*: x \mapsto \{x^* \in X^* \mid (\forall y \in X) \langle y - x, x^* \rangle + f(x) \leq f(y)\}.$$

We say a net $(a_\alpha)_{\alpha \in \Gamma}$ in X is *eventually bounded* if there exist $\alpha_0 \in \Gamma$ and $M \geq 0$ such that

$$\|a_\alpha\| \leq M, \quad \forall \alpha \succeq_\Gamma \alpha_0.$$

We denote by \longrightarrow and \rightharpoonup_{w^*} respectively, the norm convergence and weak* convergence of nets.

Let $A : X \rightrightarrows X^*$ be monotone with $\text{dom } A \neq \emptyset$ and consider a set $S \subseteq \text{dom } A$. We define $A_S : X \rightrightarrows X^*$ by

$$(3) \quad \begin{aligned} \text{gra } A_S &= \overline{\text{gra } A \cap (S \times X^*)}^{\|\cdot\| \times w^*} \\ &= \{(x, x^*) \mid \exists \text{ a net } (x_\alpha, x_\alpha^*)_{\alpha \in \Gamma} \text{ in } \text{gra } A \cap (S \times X^*) \text{ such that } x_\alpha \longrightarrow x, x_\alpha^* \rightharpoonup_{w^*} x^*\}. \end{aligned}$$

If $\text{int dom } A \neq \emptyset$, we denote by $A_{\text{int}} := A_{\text{int dom } A}$. We note that $A_{\text{dom } A} = A$ while $\text{gra } A_S \subseteq \text{gra } A_T$ for $S \subseteq T$.

Let $A : X \rightrightarrows X^*$. Following [19], we say A has the upper-semicontinuity property *property (Q)* if for every net $(x_\alpha)_{\alpha \in J}$ in X such that $x_\alpha \longrightarrow x$, we have

$$(4) \quad \bigcap_{\alpha \in J} \overline{\text{conv} \left[\bigcup_{\beta \succeq_J \alpha} A(x_\beta) \right]}^{w^*} \subseteq Ax.$$

The remainder of this paper is organized as follows. In Section 2, we collect preliminary results for future reference and the reader's convenience. In Section 3, we study local boundedness properties of monotone operators and also give a somewhat simpler proof of a recent result of Voisei [34]. The main result (Theorem 4.7) is proved in Section 4, and we also present a new proof of a result of Auslender (Theorem 4.4). A second structure theorem 4.12 — which yields a strong version of property (Q) for maximally monotone operators (Theorem 4.12 —) is also provided. Finally, in Section 5 we present a few extra examples.

2 Preliminary results

We start with a classic compactness theorem.

Fact 2.1 (Banach–Alaoglu) (See [21, Theorem 2.6.18] or [28, Theorem 3.15].) *The closed unit ball B_{X^*} in X^* is weak* compact.*

Fact 2.2 (Rockafellar) (See [26, Theorem A], [37, Theorem 3.2.8], [30, Theorem 18.7] or [13, Theorem 9.2.1].) *Let $f : X \rightarrow]-\infty, +\infty]$ be a proper lower semicontinuous convex function. Then ∂f is maximally monotone.*

The prior result can fail in both incomplete normed spaces and in complete metrizable locally convex spaces [13]. The next two important central results now has many proofs (see also [13, Ch. 8]).

Fact 2.3 (Rockafellar) (See [24, Theorem 1] or [22, Theorem 2.28].) *Let $A : X \rightrightarrows X^*$ be monotone with $\text{int dom } A \neq \emptyset$. Then A is locally bounded at $x \in \text{int dom } A$, i.e., there exist $\delta > 0$ and $K > 0$ such that*

$$\sup_{y^* \in Ay} \|y^*\| \leq K, \quad \forall y \in (x + \delta B_X) \cap \text{dom } A.$$

Fact 2.4 (Rockafellar) (See [24, Theorem 1] or [30, Theorem 27.1 and Theorem 27.3].) *Let $A : X \rightrightarrows X^*$ be maximal monotone with $\text{int dom } A \neq \emptyset$. Then $\text{int dom } A = \text{int } \overline{\text{dom } A}$ and $\overline{\text{dom } A}$ is convex.*

The final two results we give are elementary.

Fact 2.5 ([9, Section 2, page 539].) *Let $A : X \rightrightarrows X^*$ be maximally monotone and a net $(a_\alpha, a_\alpha^*)_{\alpha \in \Gamma}$ in $\text{gra } A$. Assume that $(a_\alpha, a_\alpha^*)_{\alpha \in \Gamma}$ norm \times weak* converges to (x, x^*) and $(a_\alpha^*)_{\alpha \in \Gamma}$ is eventually bounded. Then $(x, x^*) \in \text{gra } A$.*

Fact 2.6 (See [1, Proposition 4.1.7].) *Let C be a convex subset of C with $\text{int } C \neq \emptyset$. Then for every $x \in C$, $\text{int } T_C(x) = \bigcup_{\lambda > 0} \lambda [\text{int } C - x]$.*

3 Local boundedness properties

The following result is extracted from part of the proof of [35, Proposition 3.1]. For the reader's convenience, we repeat the proof here.

Fact 3.1 (Boundedness below) *Let $A : X \rightrightarrows X^*$ be monotone and $x \in \text{int dom } A$. Then there exist $\delta > 0$ and $M > 0$ such that $x + \delta B_X \subseteq \text{dom } A$ and $\sup_{a \in x + \delta B_X} \|Aa\| \leq M$. Assume that (z, z^*) is monotonically related to $\text{gra } A$. Then*

$$(5) \quad \langle z - x, z^* \rangle \geq \delta \|z^*\| - (\|z - x\| + \delta)M.$$

Proof. Since $x \in \text{int dom } A$, using Fact 2.3, there exist $\delta > 0$ and $M > 0$ such that

$$(6) \quad Aa \neq \emptyset \quad \text{and} \quad \sup_{a^* \in Aa} \|a^*\| \leq M, \quad \forall a \in (x + \delta B_X).$$

Then we have

$$\begin{aligned} & \langle z - x - b, z^* - b^* \rangle \geq 0, \quad \forall b \in \delta B_X, b^* \in A(x + b) \\ & \Rightarrow \langle z - x, z^* \rangle - \langle b, z^* \rangle + \langle z - x - b, -b^* \rangle \geq 0, \quad \forall b \in \delta B_X, b^* \in A(x + a) \\ & \Rightarrow \langle z - x, z^* \rangle - \langle b, z^* \rangle \geq \langle z - x - b, b^* \rangle, \quad \forall b \in \delta B_X, b^* \in A(x + b) \\ & \Rightarrow \langle z - x, z^* \rangle - \langle b, z^* \rangle \geq -(\|z - x\| + \delta)M, \quad \forall b \in \delta B_X \quad (\text{by (6)}) \\ (7) \quad & \Rightarrow \langle z - x, z^* \rangle \geq \langle b, z^* \rangle - (\|z - x\| + \delta)M, \quad \forall b \in \delta B_X. \end{aligned}$$

Hence we have

$$\langle z - x, z^* \rangle \geq \delta \|z^*\| - (\|z - x\| + \delta)M.$$

■

Fact 3.1 leads naturally to the following result which has many precursors.

Lemma 3.2 (Strong directional boundedness) *Let $A : X \rightrightarrows X^*$ be monotone and $x \in \text{int dom } A$. Then there exist $\delta > 0$ and $M > 0$ such that $x + 2\delta B_X \subseteq \text{dom } A$ and $\sup_{a \in x + 2\delta B_X} \|Aa\| \leq M$. Assume also that (x_0, x_0^*) is monotonically related to $\text{gra } A$. Then*

$$\sup_{a \in [x + \delta B_X, x_0[, a^* \in Aa} \|a^*\| \leq \frac{1}{\delta} (\|x_0 - x\| + 1) (\|x_0^*\| + 2M),$$

where $[x + \delta B_X, x_0[:= \{(1 - t)y + tx_0 \mid 0 \leq t < 1, y \in x + \delta B_X\}$.

Proof. Since $x \in \text{int dom } A$, by Fact 2.3, there exist $\delta > 0$ and $M > 0$ such that

$$(8) \quad x + 2\delta B_X \subseteq \text{dom } A \quad \text{and} \quad \sup_{a^* \in Aa} \|a^*\| \leq M, \quad \forall a \in (x + 2\delta B_X).$$

Let $y \in x + \delta B_X$. Then by (8),

$$(9) \quad y + \delta B_X \subseteq \text{dom } A \quad \text{and} \quad \sup_{a^* \in Aa} \|a^*\| \leq M, \quad \forall a \in (y + \delta B_X).$$

Let $t \in [0, 1[$ and $a^* \in A((1 - t)y + tx_0)$. By the assumption that (x_0, x_0^*) is monotonically related to $\text{gra } A$, we have

$$\langle a^* - x_0^*, (1 - t)(y - x_0) \rangle = \langle a^* - x_0^*, (1 - t)y + tx_0 - x_0 \rangle \geq 0.$$

Thus

$$(10) \quad \langle a^*, x_0 - y \rangle \leq \langle x_0 - y, x_0^* \rangle.$$

By Fact 3.1 and (9),

$$(11) \quad \begin{aligned} \delta \|a^*\| &\leq \langle (1 - t)y + tx_0 - y, a^* \rangle + (\|(1 - t)y + tx_0 - y\| + \delta)M \\ &\leq \langle t(x_0 - y), a^* \rangle + (\|x_0 - y\| + \delta)M \\ &\leq \langle t(x_0 - y), a^* \rangle + (\|x_0 - x\| + 2\delta)M \quad (\text{since } y \in x + \delta B_X). \end{aligned}$$

Then by (11) and (10),

$$\begin{aligned}
\|a^*\| &\leq \frac{1}{\delta} t \langle x_0 - y, x_0^* \rangle + \frac{M}{\delta} \|x_0 - x\| + 2M \leq \frac{1}{\delta} \|x_0 - y\| \cdot \|x_0^*\| + \frac{M}{\delta} \|x_0 - x\| + 2M \\
&\leq \frac{1}{\delta} (\|x_0 - x\| + \delta) \|x_0^*\| + \frac{M}{\delta} \|x_0 - x\| + 2M \quad (\text{since } y \in x + \delta B_X) \\
&\leq \frac{1}{\delta} \|x_0 - x\| \cdot \|x_0^*\| + \|x_0^*\| + \frac{M}{\delta} \|x_0 - x\| + 2M \\
&= \frac{1}{\delta} \|x_0 - x\| (\|x_0^*\| + M) + \|x_0^*\| + 2M \\
&\leq \frac{1}{\delta} (\|x_0 - x\| + 1) (\|x_0^*\| + 2M).
\end{aligned}$$

Hence

$$\sup_{a \in [x + \delta B_X, x_0], a^* \in Aa} \|a^*\| \leq \frac{1}{\delta} (\|x_0 - x\| + 1) (\|x_0^*\| + 2M).$$

We now have the required estimate. ■

The following result — originally conjectured by the first author in [7] — was established by Voisei in [34, Theorem 37] as part of a more complex set of results. We next give a somewhat simpler proof by applying a similar technique to that used in the proof of [35, Prop 3.1, subcase 2].

Theorem 3.3 (Eventual boundedness) *Let $A : X \rightrightarrows X^*$ be monotone such that $\text{int dom } A \neq \emptyset$. Then every norm \times weak* convergent net in $\text{gra } A$ is eventually bounded.*

Proof. As the result and hypotheses are again invariant under translation, we can and do suppose that $0 \in \text{int dom } A$. Let $(a_\alpha, a_\alpha^*)_{\alpha \in \Gamma}$ in $\text{gra } A$ be such that

$$(12) \quad (a_\alpha, a_\alpha^*) \text{ norm } \times \text{ weak}^* \text{ converges to } (x, x^*).$$

Clearly, it suffices to show that $(a_\alpha^*)_{\alpha \in \Gamma}$ is eventually bounded. Suppose to the contrary that $(a_\alpha^*)_{\alpha \in \Gamma}$ is not eventually bounded. Then there exists a subnet of $(a_\alpha^*)_{\alpha \in \Gamma}$, for convenience, still denoted by $(a_\alpha^*)_{\alpha \in \Gamma}$, such that

$$(13) \quad \lim_{\alpha} \|a_\alpha^*\| = +\infty.$$

We can and do suppose that $a_\alpha^* \neq 0, \forall \alpha \in \Gamma$. By Fact 3.1, there exist $\delta > 0$ and $M > 0$ such that

$$(14) \quad \langle a_\alpha, a_\alpha^* \rangle \geq \delta \|a_\alpha^*\| - (\|a_\alpha\| + \delta)M, \quad \forall \alpha \in \Gamma.$$

Then we have

$$(15) \quad \langle a_\alpha, \frac{a_\alpha^*}{\|a_\alpha^*\|} \rangle \geq \delta - \frac{(\|a_\alpha\| + \delta)M}{\|a_\alpha^*\|}, \quad \forall \alpha \in \Gamma.$$

By Fact 2.1, there exists a weak* convergent *subnet* $(a_\beta^*)_{\beta \in I}$ of $(a_\alpha^*)_{\alpha \in \Gamma}$, say

$$(16) \quad \frac{a_\beta^*}{\|a_\beta^*\|} \rightharpoonup_{w^*} a_\infty^* \in X^*.$$

Then taking the limit along the subset in (15), by (12) and (13), we have

$$(17) \quad \langle x, a_\infty^* \rangle \geq \delta.$$

On the other hand, by (12), we have

$$(18) \quad \langle x, a_\alpha^* \rangle \longrightarrow \langle x, x^* \rangle.$$

Dividing by $\|a_\alpha^*\|$ in both sides of (18), then by (13) and (16) we take the limit along the subnet again to get

$$(19) \quad \langle x, a_\infty^* \rangle = 0.$$

The above inequality contradict (17). Hence $(a_\alpha, a_\alpha^*)_{\alpha \in \Gamma}$ is eventually bounded. ■

Corollary 3.4 (Norm-weak* closed graph) *Let $A : X \rightrightarrows X^*$ be maximally monotone such that $\text{int dom } A \neq \emptyset$. Then $\text{gra } A$ is norm \times weak* closed.*

Proof. Apply Fact 2.5 and Theorem 3.3. ■

Example 3.5 (Failure of graph to be norm-weak* closed) In [9], the authors showed that the statement of Corollary 3.4 cannot hold without the assumption of the nonempty interior domain even for the subdifferential operators — actually it fails in the bw* topology. More precisely (see [9] or [4, Example 21.5]): Let $f : \ell^2(\mathbb{N}) \rightarrow]-\infty, +\infty]$ be defined by

$$(20) \quad x \mapsto \max \left\{ 1 + \langle x, e_1 \rangle, \sup_{2 \leq n \in \mathbb{N}} \langle x, \sqrt{n} e_n \rangle \right\},$$

where $e_n := (0, \dots, 0, 1, 0, \dots, 0)$: the n th entry is 1 and the others are 0. Then f is proper lower semicontinuous and convex, but ∂f is not norm \times weak* closed. A more general construction in an infinite-dimensional Banach space E is also given in [9, Section 3]. It is as follows:

Let Y be an infinite dimensional separable subspace of E , and $(v_n)_{n \in \mathbb{N}}$ be a *normalized Markushevich basis* of Y with the dual coefficients $(v_n^*)_{n \in \mathbb{N}}$. We defined $v_{p,m}$ and $v_{p,m}^*$ by

$$v_{p,m} := \frac{1}{p}(v_p + v_{p^m}) \quad \text{and} \quad v_{p,m}^* := v_p^* + (p-1)v_{p^m}^*, \quad m \in \mathbb{N}, p \text{ is prime.}$$

Let $f : E \rightarrow]-\infty, +\infty]$ be defined by

$$(21) \quad x \mapsto \iota_Y(x) + \max \{1 + \langle x, v_1^* \rangle, \sup_{2 \leq m \in \mathbb{N}, p \text{ is prime}} \langle x, v_{p,m}^* \rangle\}.$$

Then f is proper lower semicontinuous and convex. We have that ∂f is not norm \times bw* closed and hence ∂f is not norm \times weak* closed. \diamond

Corollary 3.6 *Let $A : X \rightrightarrows X^*$ be maximally monotone with $\text{int dom } A \neq \emptyset$. Assume that $S \subseteq \text{dom } A$. Then $\text{gra } A_S \subseteq \text{gra } A$ and in consequence $\overline{\text{conv } [A_S(x)]}^{\text{w}^*} \subseteq Ax, \forall x \in \text{dom } A$. Moreover, $Ax = A_S(x), \forall x \in S$ and hence $Ax = A_{\text{int}}(x), \forall x \in \text{int dom } A$.*

Proof. By (3) and Corollary 3.4, $\text{gra } A_S \subseteq \text{gra } A$. Since A is maximally monotone, (for every $x \in \text{dom } A$), Ax is convex and weak* closed. Thus $\overline{\text{conv } [A_S(x)]}^{\text{w}^*} \subseteq Ax, \forall x \in \text{dom } A$. Let $x \in S$. Then by (3) again, $Ax \subseteq A_S(x)$ and hence $Ax = A_S(x)$. Thus we have $A = A_{\text{int}}$ on $\text{int dom } A$. \blacksquare

We now turn to consequences of these boundedness results.

4 Structure of maximally monotone operators

A useful consequence of the Hahn-Banach separation principle [13] is:

Proposition 4.1 *Let D, F be nonempty subsets of X^* , and C be a convex set of X with $\text{int } C \neq \emptyset$. Assume that $x \in C$ and that for every $v \in \text{int } T_C(x)$,*

$$\sup \langle D, v \rangle \leq \sup \langle F, v \rangle < +\infty.$$

Then

$$(22) \quad D \subseteq \overline{\text{conv } F + N_C(x)}^{\text{w}^*}.$$

Proof. The separation principle ensures that suffices to show

$$(23) \quad \sup \langle D, h \rangle \leq \sup \langle N_C(x) + F, h \rangle, \quad \forall h \in X.$$

We consider two cases.

Case 1: $h \notin T_C(x)$. We have $\sup \langle N_C(x) + F, h \rangle = +\infty$ since $\sup \langle N_C(x), h \rangle = +\infty$. Hence (23) holds.

Case 2: $h \in T_C(x)$. Let $v \in \text{int } T_C(x)$. Then (for every $t > 0$) $h + tv \in \text{int } T_C(x)$ by [3, Fact 2.2(ii)]. Now $z \mapsto \sup \langle D, z \rangle$ is lower semicontinuous, and so by the assumption, we have

$$\begin{aligned} \sup \langle D, h \rangle &\leq \liminf_{t \rightarrow 0^+} \sup \langle D, h + tv \rangle \leq \liminf_{t \rightarrow 0^+} \sup \langle F, h + tv \rangle \\ &\leq \sup \langle F, h \rangle + \liminf_{t \rightarrow 0^+} t \sup \langle F, v \rangle \\ &= \sup \langle F, h \rangle \quad (\text{since } \sup \langle F, v \rangle \text{ is finite}) \\ &\leq \sup \langle N_C(x) + F, h \rangle. \end{aligned}$$

Hence (23) holds and we have (22) holds. ■

The proof of Proposition 4.1 was inspired partially by that of [3, Theorem 4.5].

We can now provide our final technical proposition.

Proposition 4.2 *Let $A : X \rightrightarrows X^*$ be maximally monotone with $S \subseteq \text{int dom } A \neq \emptyset$ such that S is dense in $\text{int dom } A$. Assume that $x \in \text{dom } A$ and $v \in H_{\overline{\text{dom } A}}(x) = \text{int } T_{\overline{\text{dom } A}}(x)$. Then there exists $x_0^* \in A_S(x)$ such that*

$$(24) \quad \sup \langle A_S(x), v \rangle = \langle x_0^*, v \rangle = \sup \langle Ax, v \rangle.$$

In particular, $\text{dom } A_S = \text{dom } A$.

Proof. By Corollary 3.6, $\text{gra } A_S \subseteq \text{gra } A$ and hence

$$(25) \quad \sup \langle A_S(x), v \rangle \leq \sup \langle Ax, v \rangle.$$

Now we show that

$$(26) \quad \sup \langle A_S(x), v \rangle \geq \sup \langle Ax, v \rangle.$$

Appealing now to Fact 2.6, we can and do suppose that $v = x_0 - x$, where $x_0 \in \overline{\text{int dom } A} = \text{int dom } A$ by Fact 2.4. Using Lemma 3.2 select $M, \delta > 0$ such that $x_0 + 2\delta B_X \subseteq \text{dom } A$ and

$$(27) \quad \sup_{a \in [x_0 + \delta B_X, x], a^* \in Aa} \|a^*\| \leq M < +\infty.$$

Let $t \in]0, 1[$. Then by Fact 2.4 again,

$$(28) \quad x + tB_\delta(v) = (1 - t)x + tx_0 + t\delta B_X \subseteq \overline{\text{int dom } A} = \text{int dom } A.$$

Then by the monotonicity of A ,

$$(29) \quad t \langle a^* - x^*, w \rangle = \langle a^* - x^*, x + tw - x \rangle \geq 0, \quad \forall a^* \in A(x + tw), x^* \in Ax, w \in B_\delta(v).$$

There exists a sequence $(x_n^*)_{n \in \mathbb{N}}$ in Ax such that

$$(30) \quad \langle x_n^*, v \rangle \longrightarrow \sup \langle Ax, v \rangle.$$

Combining (30) and (29), we have

$$(31) \quad \langle a^* - x_n^*, v + w - v \rangle \geq 0, \quad \forall a^* \in A(x + tw), w \in B_\delta(v), n \in \mathbb{N}.$$

Fix $1 < n \in \mathbb{N}$. Thus, appealing to (27) and (31) yields,

$$(32) \quad \begin{aligned} \langle a^*, v \rangle &\geq \langle x_n^*, v \rangle - \langle a^* - x_n^*, w - v \rangle \\ &\geq \langle x_n^*, v \rangle - (M + \|x_n^*\|) \cdot \|w - v\| \quad \forall a^* \in A(x + tw), w \in B_\delta(v), n \in \mathbb{N}. \end{aligned}$$

Take $\varepsilon_n := \min\{\frac{1}{n(M + \|x_n^*\|)}, \delta\}$ and $t_n := \frac{1}{n}$.

Since S is dense in $\text{int dom } A$ and $x + t_n B_{\varepsilon_n}(v) \subseteq \text{int dom } A$ by (28), $S \cap [x + t_n B_{\varepsilon_n}(v)] \neq \emptyset$. Then there exists $w_n \in X$ such that

$$(33) \quad w_n \in B_{\varepsilon_n}(v), \quad x + t_n w_n \in S \quad \text{and then} \quad x + t_n w_n \longrightarrow x.$$

Hence, by (32),

$$(34) \quad \langle a^*, v \rangle \geq \langle x_n^*, v \rangle - \frac{1}{n}, \quad \forall a^* \in A(x + t_n w_n).$$

Let $a_n^* \in A(x + t_n w_n)$. Then by (34),

$$(35) \quad \langle a_n^*, v \rangle \geq \langle x_n^*, v \rangle - \frac{1}{n}.$$

By (27) and (28), $(a_n^*)_{n \in \mathbb{N}}$ is bounded. Then by Fact 2.1, there exists a weak* convergent subnet of $(a_\alpha^*)_{\alpha \in I}$ of $(a_n^*)_{n \in \mathbb{N}}$ such that

$$(36) \quad a_\alpha^* \rightharpoonup_{w^*} x_0^* \in X^*.$$

Then by (33), $x_0^* \in A_S(x)$ and thus by (35), (36) and (30)

$$\sup \langle A_S(x), v \rangle \geq \langle x_0^*, v \rangle \geq \sup \langle Ax, v \rangle.$$

Hence (26) holds and so does (24) by (25). The final conclusion then follows from Corollary 3.6 directly. ■

An easy consequence is the reconstruction of A on the interior of its domain. In the language of [22, 23, 10, 14, 13] this is asserting the minimality of A as a w^* -cusco.

Corollary 4.3 *Let $A : X \rightrightarrows X^*$ be maximally monotone with $S \subseteq \text{int dom } A \neq \emptyset$. for any S dense in $\text{int dom } A$, we have $\overline{\text{conv}[A_S(x)]}^{w^*} = Ax = A_{\text{int}}(x), \forall x \in \text{int dom } A$.*

Proof. Corollary 3.6 shows $\text{gra } A_S \subseteq \text{gra } A$. Thus A_S is monotone. By proposition 4.2, $A_S(x) \neq \emptyset$ on $\text{dom } A$. Then apply [22, Theorem 7.13 and Corollary 7.8] and Corollary 3.6 to obtain

$$\overline{\text{conv } [A_S(x)]}^{w^*} = Ax = A_{\text{int}}(x), \quad \forall x \in \text{int dom } A,$$

as required. ■

There are many possible extensions of this sort of result along the lines studied in [10]. Applying Proposition 4.2 and Lemma 3.2, we can also quickly recapture [2, Theorem 2.1].

Theorem 4.4 (Directional boundedness in Euclidean space) *Suppose that X is finite-dimensional. Let $A : X \rightrightarrows X^*$ be maximally monotone and $x \in \text{dom } A$. Assume that there exist $d \in X$ and $\varepsilon_0 > 0$ such that $x + \varepsilon_0 d \in \text{int dom } A$. Then*

$$[Ax]_d := \{x^* \in Ax \mid \langle x^*, d \rangle = \sup \langle Ax, d \rangle\}$$

is nonempty and compact. Moreover, if a sequence $(x_n)_{n \in \mathbb{N}}$ in $\text{dom } A$ is such that $x_n \rightarrow x$ and

$$(37) \quad \lim \frac{x_n - x}{\|x_n - x\|} = d,$$

then for every $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that

$$(38) \quad A(x_n) \subseteq [Ax]_d + \varepsilon B_{X^*}, \quad \forall n \geq N.$$

Proof. By Fact 2.6, we have $d = \frac{1}{\varepsilon_0}(x + \varepsilon_0 d - x) \in \frac{1}{\varepsilon_0}[\text{int dom } A - x] \subseteq \text{int } T_{\overline{\text{dom } A}}(x)$. Then by Proposition 4.2 and Corollary 3.6, there exists $v^* \in Ax$ such that

$$(39) \quad \sup \langle Ax, d \rangle = \langle v^*, d \rangle.$$

Hence $v^* \in [Ax]_d$ and thus $[Ax]_d \neq \emptyset$.

We next show that $[Ax]_d$ is compact. Let $x^* \in [Ax]_d$. By Fact 3.1, there exist $\delta > 0$ and $M > 0$ such that $-\varepsilon_0 \langle d, x^* \rangle = \langle x - (x + \varepsilon_0 d), x^* \rangle \geq \delta \|x^*\| - (\|\varepsilon_0 d\| + \delta)M$. Then by (39), $\delta \|x^*\| \leq (\|\varepsilon_0 d\| + \delta)M - \varepsilon_0 \langle d, x^* \rangle = (\|\varepsilon_0 d\| + \delta)M - \varepsilon_0 \langle d, v^* \rangle < +\infty$. Hence $[Ax]_d$ is bounded. Clearly, $[Ax]_d$ is closed and so $[Ax]_d$ is compact.

Finally, we show that (38) holds. By Lemma 3.2 and $x + \varepsilon_0 d \in \text{int dom } A$, there exists $\delta_1 > 0$ such that

$$(40) \quad \sup_{a \in [x + \varepsilon_0 d + \delta_1 B_X, x], a^* \in Aa} \|a^*\| < +\infty.$$

By (37), we have $\|d\| = 1$ and there exists $N \in \mathbb{N}$ such that for every $n \geq N$, $0 < \|x_n - x\| < \varepsilon_0$ and $x_n \in x + \|x_n - x\|d + \|x_n - x\|\frac{\delta_1}{\varepsilon_0}B_X \subseteq [x + \varepsilon_0d + \delta_1B_X, x[$. Then by (40),

$$(41) \quad \sup_{a^* \in A(x_n), n \geq N} \|a^*\| < +\infty.$$

Suppose to the contrary that (38) does not holds. Then there exists $\varepsilon_1 > 0$ and a subsequence $(x_{n,k})_{k \in \mathbb{N}}$ of $(x_n)_{n \in \mathbb{N}}$ and $x_{n,k}^* \in A(x_{n,k})$ such that

$$(42) \quad x_{n,k}^* \notin [Ax]_d + \varepsilon_1 B_{X^*}, \quad \forall k \in \mathbb{N}.$$

By (41), there exists a convergent subsequence of $(x_{n,k}^*)_{k \in \mathbb{N}}$, for convenience, still denoted by $(x_{n,k}^*)_{k \in \mathbb{N}}$ such that

$$(43) \quad x_{n,k}^* \longrightarrow x_\infty^*.$$

Since $x_{n,k} \longrightarrow x$, by (43),

$$(44) \quad (x, x_\infty^*) \in \text{gra } A.$$

We claim that

$$(45) \quad x_\infty^* \in [Ax]_d.$$

By the monotonicity of A , recalling (39), we have $\langle x_{n,k}^* - v^*, x_{n,k} - x \rangle \geq 0$, $\forall k \in \mathbb{N}$. Hence

$$(46) \quad \langle x_{n,k}^* - v^*, \frac{x_{n,k} - x}{\|x_{n,k} - x\|} \rangle \geq 0, \quad \forall k \in \mathbb{N}.$$

Combining (43), (37) and (46),

$$(47) \quad \langle x_\infty^* - v^*, d \rangle \geq 0.$$

By (39), (47) and (44), $x_\infty^* \in [Ax]_d$ and hence (45) holds. Then $x_\infty^* + \varepsilon_1 B_X \subseteq [Ax]_d + \varepsilon_1 B_X$ and $x_\infty^* + \varepsilon_1 B_X$ contains infinitely many terms of $(x_{n,k}^*)_{k \in \mathbb{N}}$, which contradicts (42). Hence, (38) holds as asserted. \blacksquare

Remark 4.5 In the statement of [2, Theorem 2.1], the “ $x - x_n$ ” in Eq (2.0) should be read as “ $x_n - x$ ”. In his proof, the author considered it as “ $x_n - x$ ”. \diamond

We next recall an alternate *recession cone* description of $N_{\text{dom } A}$. Consider

$$(48) \quad \text{rec } A(x) := \{x^* \in X^* \mid \exists t_n \rightarrow 0^+, (a_n, a_n^*) \in \text{gra } A \text{ such that } a_n \longrightarrow x, t_n a_n \rightharpoonup_{w^*} x^*\}.$$

Proposition 4.6 (Recession cone) *Let $A : X \rightrightarrows X^*$ be monotone with $\text{gra } A \neq \emptyset$. Then for every $x \in \text{dom } A$ one has*

$$N_{\overline{\text{dom } A}}(x) = \text{rec } A(x).$$

Proof. Let $x \in \text{dom } A$. We first show that

$$(49) \quad \text{rec } A(x) \subseteq N_{\overline{\text{dom } A}}(x).$$

Let $x^* \in \text{rec } A(x)$. Then there exist $(t_n)_{n \in \mathbb{N}}$ in \mathbb{R} and $(a_n, a_n^*)_{n \in \mathbb{N}}$ in $\text{gra } A$ such that

$$(50) \quad t_n \longrightarrow 0^+, \quad a_n \longrightarrow x \quad \text{and} \quad t_n a_n^* \rightharpoonup_{w^*} x^*.$$

By [21, Corollary 2.6.10], $(t_n a_n^*)_{n \in \mathbb{N}}$ is bounded. By the monotonicity of A ,

$$\langle a_n - a, a_n^* \rangle \geq \langle a_n - a, a^* \rangle, \quad \forall (a, a^*) \in \text{gra } A.$$

Therefore,

$$(51) \quad \langle a_n - a, t_n a_n^* \rangle \geq t_n \langle a_n - a, a^* \rangle, \quad \forall (a, a^*) \in \text{gra } A.$$

Taking the limit in (51), by (50), we have

$$\langle x - a, x^* \rangle \geq 0, \quad \forall a \in \text{dom } A.$$

Thus, $x^* \in N_{\overline{\text{dom } A}}(x)$. Hence (49) holds.

It remains to show that

$$(52) \quad N_{\overline{\text{dom } A}}(x) \subseteq \text{rec } A(x).$$

Let $y^* \in N_{\overline{\text{dom } A}}(x)$ and $n \in \mathbb{N}$. Take $v^* \in Ax$. Since $A = N_{\overline{\text{dom } A}} + A$, we have $ny^* + v^* \in Ax$. Set $a_n := x$, $a_n^* := ny^* + v^*$ and $t_n := \frac{1}{n}$. Then we have

$$a_n \longrightarrow x, \quad t_n \longrightarrow 0^+ \quad \text{and} \quad t_n a_n^* = y^* + \frac{1}{n} v^* \longrightarrow y^*.$$

Hence $y^* \in \text{rec } A(x)$ and then (52) holds.

Combining (49) and (52), we have $N_{\overline{\text{dom } A}}(x) = \text{rec } A(x)$. ■

We are now ready for our main result, Theorem 4.7, the proof of which was inspired partially by that of [36, Theorem 3.1].

Theorem 4.7 (Reconstruction of A , I) *Let $A : X \rightrightarrows X^*$ be maximally monotone with $S \subseteq \text{int dom } A \neq \emptyset$ and with S dense in $\text{int dom } A$. Then*

$$(53) \quad Ax = N_{\overline{\text{dom } A}}(x) + \overline{\text{conv}[A_S(x)]}^{w*} = \text{rec } A(x) + \overline{\text{conv}[A_S(x)]}^{w*}, \quad \forall x \in X,$$

where $\text{rec } A(x)$ is as in (48).

Proof. We first show that

$$(54) \quad Ax = \overline{N_{\overline{\text{dom } A}}(x) + \text{conv}[A_S(x)]}^{w*}, \quad \forall x \in X.$$

By Corollary 3.6, we have $\text{conv}[A_S(x)] \subseteq Ax$, $\forall x \in X$. Since likewise $A = A + N_{\overline{\text{dom } A}}$,

$$(55) \quad \overline{N_{\overline{\text{dom } A}}(x) + \text{conv}[A_S(x)]}^{w*} \subseteq Ax, \quad \forall x \in X.$$

It remains show that

$$(56) \quad Ax \subseteq \overline{N_{\overline{\text{dom } A}}(x) + \text{conv}[A_S(x)]}^{w*}, \quad \forall x \in \text{dom } A.$$

Let $x \in \text{dom } A$. By the maximal monotonicity of A and Proposition 4.2, both Ax and $A_S(x)$ are nonempty sets. Then applying Proposition 4.1 and Proposition 4.2 directly, we have (56) holds and hence (54) holds.

We must still show

$$(57) \quad Ax = N_{\overline{\text{dom } A}}(x) + \overline{\text{conv}[A_S(x)]}^{w*}, \quad \forall x \in X.$$

Now, for every two sets $C, D \subseteq X^*$, we have $C + \overline{D}^{w*} \subseteq \overline{C + D}^{w*}$. Then by (54), it suffices to show that

$$(58) \quad \overline{N_{\overline{\text{dom } A}}(x) + \text{conv}[A_S(x)]}^{w*} \subseteq N_{\overline{\text{dom } A}}(x) + \overline{\text{conv}[A_S(x)]}^{w*}, \quad \forall x \in \text{dom } A.$$

We again can and do suppose that $0 \in \text{int dom } A$ and $(0, 0) \in \text{gra } A$. Let $x \in \text{dom } A$ and $x^* \in \overline{N_{\overline{\text{dom } A}}(x) + \text{conv}[A_S(x)]}^{w*}$. Then there exists nets $(x_\alpha^*)_{\alpha \in I}$ in $N_{\overline{\text{dom } A}}(x)$ and $(y_\alpha^*)_{\alpha \in I}$ in $\text{conv}[A_S(x)]$ such that

$$(59) \quad x_\alpha^* + y_\alpha^* \rightharpoonup_{w*} x^*.$$

Now we claim that

$$(60) \quad (x_\alpha^*)_{\alpha \in I} \text{ is eventually bounded.}$$

Suppose to the contrary that $(x_\alpha^*)_{\alpha \in I}$ is not eventually bounded. Then there exists a subnet of $(x_\alpha^*)_{\alpha \in I}$, for convenience, still denoted by $(x_\alpha^*)_{\alpha \in I}$, such that

$$(61) \quad \lim_{\alpha} \|x_\alpha^*\| = +\infty.$$

We can and do suppose that $x_\alpha^* \neq 0, \forall \alpha \in I$. By $0 \in \text{int dom } A$ and $x_\alpha^* \in N_{\overline{\text{dom } A}}(x)$ (for every $\alpha \in I$), there exists $\delta > 0$ such that $\delta B_X \subseteq \overline{\text{dom } A}$ and hence we have

$$(62) \quad \langle x, x_\alpha^* \rangle \geq \sup_{b \in B_X} \langle x_\alpha^*, \delta b \rangle = \delta \|x_\alpha^*\|.$$

Thence, we have

$$(63) \quad \langle x, \frac{x_\alpha^*}{\|x_\alpha^*\|} \rangle \geq \delta.$$

By Fact 2.1, there exists a weak* convergent subnet $(x_\beta^*)_{\beta \in \Gamma}$ of $(x_\alpha^*)_{\alpha \in I}$, say

$$(64) \quad \frac{x_\beta^*}{\|x_\beta^*\|} \rightharpoonup_{w^*} x_\infty^* \in X^*.$$

Taking the limit along the subnet in (63), by (64), we have

$$(65) \quad \langle x, x_\infty^* \rangle \geq \delta.$$

By (59) and (61), we have

$$(66) \quad \frac{x_\alpha^*}{\|x_\alpha^*\|} + \frac{y_\alpha^*}{\|x_\alpha^*\|} \rightharpoonup_{w^*} 0.$$

And so by (64),

$$(67) \quad \frac{y_\beta^*}{\|x_\beta^*\|} \rightharpoonup_{w^*} -x_\infty^*.$$

By Corollary 3.6, $\text{conv}[A_S(x)] \subseteq Ax$, and hence $(y_\alpha^*)_{\alpha \in I}$ is in Ax . Since $(0, 0) \in \text{gra } A$, we have $\langle y_\alpha^*, x \rangle \geq 0$ and so

$$(68) \quad \langle \frac{y_\beta^*}{\|x_\beta^*\|}, x \rangle \geq 0.$$

Using (67) and taking the limit along the subnet in (68) we get

$$(69) \quad \langle -x_\infty^*, x \rangle \geq 0,$$

which contradicts (65). Hence, $(x_\alpha^*)_{\alpha \in I}$ is eventually bounded and thus (60) holds.

Then by Fact 2.1 again, there exists a weak* convergent subset of $(x_\alpha^*)_{\alpha \in I}$, for convenience, still denoted by $(x_\alpha^*)_{\alpha \in I}$ which lies in the normal cone, such that $x_\alpha^* \rightharpoonup_{w^*} w^* \in X^*$. Hence $w^* \in N_{\overline{\text{dom } A}}(x)$ and $y_\alpha^* \rightharpoonup_{w^*} x^* - w^* \in \overline{\text{conv}[A_S(x)]^{w^*}}$ by (59). Hence $x^* \in N_{\overline{\text{dom } A}}(x) + \overline{\text{conv}[A_S(x)]^{w^*}}$ so that (58) holds. Then we apply Proposition 4.6 to get (53) directly. ■

Remark 4.8 If X is a *weak Asplund space* (as holds if X has a Gâteaux smooth equivalent norm, see [22, 23, 10]), the nets defined in A_S in Proposition 4.2 and Theorem 4.7 can be replaced by sequences. By [17, Chap. XIII, Notes and Remarks, page 239], B_{X^*} is weak* sequentially compact. In fact, see [13, Chpt. 9], this holds somewhat more generally.

Hence, throughout the proof of Proposition 4.2, we can obtain weak* convergent subsequences instead of subnets. The rest of each subsequent argument is unchanged. \diamond

In various classes of Banach space we can choose useful structure for $S \in S_A$, where

$$S_A := \{S \subseteq \text{int dom } A \mid S \text{ is dense in int dom } A\}.$$

Corollary 4.9 (Specification of S_A) *Let $A : X \rightrightarrows X^*$ be maximally monotone with $\text{int dom } A \neq \emptyset$. We may choose the dense set $S \in S_A$ to be as follows:*

- (i) *In a Gâteaux smooth space, entirely within the residual set of non- σ porous points of $\text{dom } A$,*
- (ii) *In an Asplund space, to include only a subset of the generic set points of single-valuedness and norm to norm continuity of A ,*
- (iii) *In a separable Asplund space, to hold only countably many angle-bounded points of A ,*
- (iv) *In a weak Asplund space, to include only a subset of the generic set of points of single-valuedness (and norm to weak* continuity) of A ,*
- (v) *In a separable space, to include only points of single-valuedness (and norm to weak* continuity) of A whose complement is covered by a countable union of Lipschitz surfaces.*
- (vi) *In finite dimensions, to include only points of differentiability of A which are of full measure.*

Proof. It suffices to determine in each case that the points of the given kind are dense. (i): See [18, Theorem 5.1]. (ii): See [22, Lemma 2.18 and Theorem 2.30]. (iii): See [22, Theorem 2.19 and Theorem 2.11]. (iv): See [23, Proposition 1.1(iii) and Theorem 1.6] or [22, Theorem 4.31 and Example 7.2]. (v): See [32, 33]. (vi): See [27, Corollary 12.66(a)] or [13, Exercise 9.1.1(2), page 412]. \blacksquare

These classes are sufficient but not necessary: for example, there are Asplund spaces with no equivalent Gâteaux smooth renorm [13]. Note also that in (v) and (vi) we also know that $A \setminus S$ is a null set in the senses discussed [16].

We now restrict attention to convex functions.

Corollary 4.10 (Convex subgradients) *Let $f : X \rightarrow]-\infty, +\infty]$ be proper lower semi-continuous and convex with $\text{int dom } f \neq \emptyset$. Let $S \subseteq \text{int dom } f$ be given with S dense in $\text{dom } f$. Then*

$$\partial f(x) = N_{\overline{\text{dom } f}}(x) + \overline{\text{conv}[(\partial f)_S(x)]}^{w*} = N_{\text{dom } f}(x) + \overline{\text{conv}[(\partial f)_S(x)]}^{w*}, \quad \forall x \in X.$$

Proof. By [22, Proposition 3.3 and Proposition 1.11], $\text{int dom } \partial f \neq \emptyset$. By the Brøndsted-Rockafellar Theorem (see [22, Theorem 3.17] or [37, Theorem 3.1.2]), $\overline{\text{dom } \partial f} = \overline{\text{dom } f}$. Then we may apply Fact 2.2 and Theorem 4.7 to get (for every $x \in X$) $\partial f(x) = N_{\overline{\text{dom } f}}(x) + \overline{\text{conv}[(\partial f)_S(x)]}^{w*}$. We have $N_{\overline{\text{dom } f}}(x) = N_{\text{dom } f}(x), \forall x \in \text{dom } \partial f$. Hence $\partial f(x) = N_{\text{dom } f}(x) + \overline{\text{conv}[(\partial f)_S(x)]}^{w*}, \forall x \in X$. ■

In this case Corollary 4.9 specifies settings in which only points of differentiability need be used (in (vi) we recover Alexandroff's theorem on twice differentiability of convex functions), see [13] for more details.

Remark 4.11 Results closely related to Corollary 4.10 have been obtained in [25, 3, 20, 31] and elsewhere. Interestingly, in the convex case we have obtained as much information more easily than by the direct convex analysis approach of [3]. ◇

We finish this section by refining Corollary 4.3 and Theorem 4.7.

Let $A : X \rightrightarrows X^*$. We define $\widehat{A} : X \rightrightarrows X^*$ by

$$(70) \quad \text{gra } \widehat{A} := \{(x, x^*) \in X \times X^* \mid x^* \in \bigcap_{\varepsilon > 0} \overline{\text{conv}[A(x + \varepsilon B_X)]}^{w*}\}.$$

Clearly, we have $\overline{\text{gra } A}^{\|\cdot\| \times w*} \subseteq \text{gra } \widehat{A}$.

Theorem 4.12 (Reconstruction of A , II) *Let $A : X \rightrightarrows X^*$ be maximally monotone with $\text{int dom } A \neq \emptyset$.*

(i) *Then $\widehat{A} = A$.*

In particular, A has property (Q); and so has a norm \times weak closed graph.*

(ii) *Moreover, if $S \subseteq \text{int dom } A$ is dense in $\text{int dom } A$ then*

$$(71) \quad \widehat{A}_S(x) := \bigcap_{\varepsilon > 0} \overline{\text{conv}[A(S \cap (x + \varepsilon B_X))]}^{w*} \supseteq \overline{\text{conv}[A_S(x)]}^{w*}, \quad \forall x \in X.$$

Thence

$$(72) \quad Ax = \widehat{A}_S(x) + \text{rec } A(x), \quad \forall x \in X.$$

Proof. Part (i). We first show that $\text{gra } \widehat{A} \subseteq \text{gra } A$. Let $(x, x^*) \in \text{gra } \widehat{A}$. Now we show that $x \in \text{dom } A$. We suppose that $0 \in \text{int dom } A$. Since $x^* \in \overline{\text{conv} [A(x + \frac{1}{n}B_X)]}^{\text{w}^*}$ (for all $n \in \mathbb{N}$),

$$\begin{aligned} \inf \langle A(x + \frac{1}{n}B_X), x \rangle &= \inf \langle \text{conv} [A(x + \frac{1}{n}B_X)], x \rangle = \inf \langle \overline{\text{conv} [A(x + \frac{1}{n}B_X)]}^{\text{w}^*}, x \rangle \\ &< \langle x, x^* \rangle + 1. \end{aligned}$$

Then there exists $z_n^* \in A(z_n)$ such that

$$(73) \quad \langle z_n^*, x \rangle \leq \langle x^*, x \rangle + 1,$$

where $z_n \in x + \frac{1}{n}B_X$. By Fact 3.1, there exist $\delta_0 > 0$ and $M_0 > 0$ such that

$$\begin{aligned} \delta_0 \|z_n^*\| &\leq \langle z_n, z_n^* \rangle + (\|z_n\| + \delta)M_0 = \langle z_n - x, z_n^* \rangle + \langle x, z_n^* \rangle + (\|z_n\| + \delta)M_0 \\ &\leq \frac{1}{n} \|z_n^*\| + \langle x^*, x \rangle + 1 + (\|x\| + 1 + \delta)M_0, \quad \forall n \in \mathbb{N} \quad (\text{by (73)}). \end{aligned}$$

Hence $(z_n^*)_{n \in \mathbb{N}}$ is bounded. By Fact 2.1, there exists a weak* convergent limit z_∞^* of a subnet of $(z_n^*)_{n \in \mathbb{N}}$. Then $z_n \rightarrow x$ and the maximal monotonicity of A , imply that $(x, z_\infty^*) \in \text{gra } A$ and so $x \in \text{dom } A$.

Now let $v \in \text{int } T_{\overline{\text{dom } A}}(x)$. We claim that

$$(74) \quad \sup \langle \widehat{A}(x), v \rangle \leq \sup \langle Ax, v \rangle.$$

By Fact 2.6, we can and do suppose that $v = x_0 - x$, where $x_0 \in \text{int } \overline{\text{dom } A} = \text{int dom } A$ by Fact 2.4. There exists a sequence $(y_n^*)_{n \in \mathbb{N}}$ in $\widehat{A}x$ such that

$$(75) \quad \langle y_n^*, v \rangle \rightarrow \sup \langle \widehat{A}x, v \rangle.$$

Using Lemma 3.2 select $M, \delta > 0$ such that $x_0 + 2\delta B_X \subseteq \text{dom } A$ and

$$(76) \quad \sup_{a \in [x_0 + \delta B_X, x[, a^* \in Aa} \|a^*\| \leq M < +\infty.$$

Then by Fact 2.4 again,

$$(77) \quad [x_0 + \delta B_X, x[\subseteq \text{int } \overline{\text{dom } A} = \text{int dom } A.$$

Fix $\frac{1}{\delta} < n \in \mathbb{N}$. Since $y_n^* \in \overline{\text{conv} [A(x + \frac{1}{n}B_X)]}^{\text{w}^*}$, then $\langle y_n^*, v \rangle \leq \sup \langle A(x + \frac{1}{n}B_X), v \rangle$. Then there exist $x_n \in (x + \frac{1}{n}B_X)$ and $x_n^* \in A(x_n)$ such that

$$(78) \quad \langle x_n^*, v \rangle \geq \langle y_n^*, v \rangle - \frac{1}{n}.$$

Set $t_n := \frac{1}{\delta n}$. Then,

$$(79) \quad \begin{aligned} a_n &:= x_n + t_n v = x_n - x + x + t_n(x_0 - x) = t_n \left(x_0 + \frac{x_n - x}{t_n} \right) + (1 - t_n)x \\ &\in t_n(x_0 + \delta B_X) + (1 - t_n)x. \end{aligned}$$

Select $a_n^* \in A(a_n)$ by (77). Then by the monotonicity of A , $t_n \langle a_n^* - x_n^*, v \rangle = \langle a_n^* - x_n^*, a_n - x_n \rangle \geq 0$. Hence $\langle a_n^*, v \rangle \geq \langle x_n^*, v \rangle$. Using (78), we have

$$(80) \quad \langle a_n^*, v \rangle \geq \langle y_n^*, v \rangle - \frac{1}{n}, \quad \forall \frac{1}{\delta} < n \in \mathbb{N}.$$

Thus, appealing to (76) and (79) shows that $(a_n^*)_{n \in \mathbb{N}}$ is bounded. Fact 2.1, now yields a weak* convergent subnet of $(a_\alpha^*)_{\alpha \in I}$ of $(a_n^*)_{n \in \mathbb{N}}$ such that

$$(81) \quad a_\alpha^* \rightharpoonup_{w^*} x_0^* \in X^*.$$

By Corollary 3.4 and $a_n \rightarrow x$, we have $x_0^* \in Ax$. Combining (80), (81) and (75), we obtain

$$\sup \langle Ax, v \rangle \geq \langle x_0^*, v \rangle \geq \sup \langle \widehat{Ax}, v \rangle.$$

Hence (74) holds. Now applying Proposition 4.1 and Proposition 4.2, we have $\widehat{Ax} \subseteq \overline{Ax + N_{\text{dom } A}(x)}^{w^*} = Ax$. Hence $\text{gra } \widehat{A} \subseteq \text{gra } A$.

Since $\text{gra } A \subseteq \text{gra } \widehat{A}$, we have $\widehat{A} = A$. It is immediate A has property (Q) so has a norm \times weak* closed graph.

Part (ii). It only remains to prove (72). We first show that

$$(82) \quad A_S(x) \subseteq \widehat{A_S}(x), \quad \forall x \in X.$$

By Proposition 4.2, $\text{dom } A_S = \text{dom } A$. Let $w \in X$. If $w \notin \text{dom } A$, then clearly, $A_S(w) \subseteq \widehat{A_S}(w)$. Assume that $w \in \text{dom } A$ and $w^* \in A_S(w)$. Then by (3), there exist a net $(w_\alpha, w_\alpha^*)_{\alpha \in I}$ in $\text{gra } A \cap (S \times X^*)$ such that $w_\alpha \rightarrow w$ and $w_\alpha^* \rightharpoonup_{w^*} w^*$. The for every $\varepsilon > 0$, there exists $\alpha_0 \in I$ such that $w_\alpha \in x + \varepsilon B_X$, $\forall \alpha \succeq_I \alpha_0$. Thus

$$w_\alpha \in S \cap (w + \varepsilon B_X) \quad \text{and then} \quad w_\alpha^* \in A(S \cap (w + \varepsilon B_X)), \quad \forall \alpha \succeq_I \alpha_0.$$

Hence $w^* \in \overline{A(S \cap (w + \varepsilon B_X))}^{w^*} \subseteq \overline{\text{conv}[A(S \cap (w + \varepsilon B_X))]}^{w^*}$ and thus (82) holds.

By (82), we have

$$(83) \quad \overline{\text{conv}[A_S(x)]}^{w^*} \subseteq \widehat{A_S}(x), \quad \forall x \in X.$$

Then by Proposition 4.6,

$$\overline{\text{conv}[A_S(x)]}^{w^*} + \text{rec } A(x) \subseteq \widehat{A_S}(x) + \text{rec } A(x) \subseteq Ax + \text{rec } A(x) = Ax, \quad \forall x \in X.$$

Thus, on appealing to Theorem 4.7, we obtain (72). ■

Remark 4.13 Property (Q,) first introduced by Cesari in Euclidean space, was recently established for maximally monotone operators with nonempty domain interior in Banach space by Voisei in [34, Theorem 42].

5 Final examples and applications

In general, we do not have $Ax = \overline{\text{conv}[A_S(x)]}^{w*}, \forall x \in \text{dom } A$, for a maximally monotone operator $A : X \rightrightarrows X^*$ with $S \subseteq \text{int dom } A \neq \emptyset$ such that S is dense in $\text{dom } A$.

We give a simple example to demonstrate this.

Example 5.1 Let C be a closed convex subset of X with $S \subseteq \text{int } C \neq \emptyset$ such that S is dense in C . Then N_C is maximally monotone and $\text{gra}(N_C)_S = C \times \{0\}$, but $N_C(x) \neq \overline{\text{conv}[(N_C)_S(x)]}^{w*}, \forall x \in \text{bdry } C$. We have $\bigcap_{\varepsilon > 0} \overline{\text{conv}[N_C(x + \varepsilon B_X)]}^{w*} = N_C(x), \forall x \in X$.

Proof. The maximal monotonicity of N_C is directly from Fact 2.2. Since, for every $x \in \text{int } C$, $N_C(x) = \{0\}$, $\text{gra}(N_C)_S = C \times \{0\}$ by (3) and Proposition 4.2. Hence $\overline{\text{conv}[(N_C)_S(x)]}^{w*} = \{0\}, \forall x \in C$. However, $N_C(x)$ is unbounded, $\forall x \in \text{bdry } C$. Hence $N_C(x) \neq \overline{\text{conv}[(N_C)_S(x)]}^{w*}, \forall x \in \text{bdry } C$.

By contrast, on applying Theorem 4.12, we have $\bigcap_{\varepsilon > 0} \overline{\text{conv}[N_C(x + \varepsilon B_X)]}^{w*} = N_C(x), \forall x \in X$. ■

While the subdifferential operators in Example 3.5 necessarily fail to have property (Q), it is possible for operators with no points of continuity to possess the property. Considering any closed linear mapping A from a reflexive space X to its dual, we have $\widehat{A} = A$ and hence A has property (Q). More generally:

Example 5.2 Suppose that X is reflexive. Let $A : X \rightrightarrows X^*$ be such that $\text{gra } A$ is nonempty closed and convex. Then $\widehat{A} = A$ and hence A has property (Q).

Proof. It suffices to show that $\text{gra } \widehat{A} \subseteq \text{gra } A$. Let $(x, x^*) \in \text{gra } \widehat{A}$. Then we have

$$x^* \in \bigcap_{n \in \mathbb{N}} \overline{\text{conv}\left[A\left(x + \frac{1}{n}B_X\right)\right]}^{w*} = \bigcap_{n \in \mathbb{N}} \overline{\text{conv}\left[A\left(x + \frac{1}{n}B_X\right)\right]} = \bigcap_{n \in \mathbb{N}} \overline{A\left(x + \frac{1}{n}B_X\right)}.$$

Then there exists a sequence $(a_n, a_n^*)_{n \in \mathbb{N}}$ in $\text{gra } A$ such that $a_n \rightarrow x, a_n^* \rightarrow x^*$. The closedness of $\text{gra } A$ implies that $(x, x^*) \in \text{gra } A$. Then $\text{gra } \widehat{A} \subseteq \text{gra } A$. ■

It would be interesting to know whether \widehat{A} and A can differ for a maximal operator with norm \times weak* closed graph.

Finally, we illustrate what Corollary 4.10 says in the case of $x \mapsto \iota_{B_X}(x) + \frac{1}{p}\|x\|^p$.

Example 5.3 Let $p > 1$ and $f : X \rightarrow]-\infty, +\infty]$ be defined by

$$x \mapsto \iota_{B_X}(x) + \frac{1}{p}\|x\|^p.$$

Then for every $x \in \text{dom } f$, we have

$$(84) \quad N_{\text{dom } f}(x) = \begin{cases} \mathbb{R}_+ \cdot Jx, & \text{if } \|x\| = 1; \\ \{0\}, & \text{if } \|x\| < 1 \end{cases}$$

$$(85) \quad (\partial f)_{\text{int}}(x) = \begin{cases} \|x\|^{p-2} \cdot Jx, & \text{if } \|x\| \neq 0; \\ \{0\}, & \text{otherwise} \end{cases}$$

where $J := \partial \frac{1}{2}\|\cdot\|^2$ and $\mathbb{R}_+ := [0, +\infty[$. Moreover, $\partial f = N_{\text{dom } f} + (\partial f)_{\text{int}} = N_{\text{dom } f} + \partial \frac{1}{p}\|\cdot\|^p$, and then $\partial f(x) \neq (\partial f)_{\text{int}}(x) = \overline{\text{conv}[(\partial f)_{\text{int}}(x)]}^{w^*}$, $\forall x \in \text{bdry dom } f$. We also have $\bigcap_{\varepsilon > 0} \overline{\text{conv}[\partial f(x + \varepsilon B_X)]}^{w^*} = \partial f(x)$, $\forall x \in X$.

Proof. By Fact 2.2, ∂f is maximally monotone. We have

$$(86) \quad \partial f = \partial \frac{1}{p}\|\cdot\|^p, \quad \forall x \in \text{int dom } \partial f.$$

By [3, Lemma 6.2],

$$(87) \quad \partial \frac{1}{p}\|\cdot\|^p(x) = \begin{cases} \|x\|^{p-2} \cdot Jx, & \text{if } \|x\| \neq 0; \\ \{0\}, & \text{otherwise.} \end{cases}$$

Now we show that

$$(88) \quad (\partial f)_{\text{int}}(x) = \partial \frac{1}{p}\|\cdot\|^p(x), \quad \forall x \in \text{dom } f.$$

Let $x \in \text{dom } f$. By Corollary 3.4 and (86), we have

$$(89) \quad (\partial f)_{\text{int}}(x) \subseteq \partial \frac{1}{p}\|\cdot\|^p(x).$$

Let $x^* \in \partial \frac{1}{p}\|\cdot\|^p(x)$. We first show that $(x, x^*) \in \text{gra}(\partial f)_{\text{int}}$. If $\|x\| < 1$, then $x \in \text{int dom } f$ and hence by (86) and Corollary 3.6, $x^* \in \partial f(x) = (\partial f)_{\text{int}}(x)$. Now we suppose that $\|x\| = 1$.

By (87), $x^* \in Jx$. Then $\frac{n-1}{n}x^* \in J(\frac{n-1}{n}x)$ and hence $(\frac{n-1}{n})^{p-1}x^* \in \partial_p^1 \|\cdot\|^p(\frac{n-1}{n}x)$ by (87), $\forall n \in \mathbb{N}$. By (86),

$$(90) \quad (\frac{n-1}{n})^{p-1}x^* \in \partial f(\frac{n-1}{n}x), \quad \forall n \in \mathbb{N}.$$

Since $0 \in \text{int dom } f$, $\frac{n-1}{n}x \in \text{int dom } f = \text{int dom } \partial f, \forall n \in \mathbb{N}$. Since $\frac{n-1}{n}x \rightarrow x, (\frac{n-1}{n})^{p-1}x^* \rightarrow x^*$, by (90), $x^* \in (\partial f)_{\text{int}}(x)$. Hence $\partial_p^1 \|\cdot\|^p(x) \subseteq (\partial f)_{\text{int}}(x)$. Thus by (89), we have (88) holds and then we obtain (85) by (87).

By (88),

$$(91) \quad (\partial f)_{\text{int}}(x) = \overline{\text{conv}[(\partial f)_{\text{int}}(x)]}^{w^*}, \quad \forall x \in \text{dom } f.$$

On the other hand, since $N_{\text{dom } f} = N_{B_X}$, we can immediately get (84).

Then by Corollary 4.10, (91) and (88), we have

$$(92) \quad \partial f(x) = N_{\text{dom } f}(x) + (\partial f)_{\text{int}}(x) = N_{\text{dom } f}(x) + \partial_p^1 \|\cdot\|^p(x), \quad \forall x \in X.$$

Let $x \in \text{bdry dom } f$. Then $\|x\| = 1$. On combining (92), (84) and (85),

$$\partial f(x) = [1, +\infty[\cdot Jx \neq Jx = (\partial f)_{\text{int}}(x) = \overline{\text{conv}[(\partial f)_{\text{int}}(x)]}^{w^*}.$$

Theorem 4.12 again implies that $\bigcap_{\varepsilon > 0} \overline{\text{conv}[\partial f(x + \varepsilon B_X)]}^{w^*} = \partial f(x), \forall x \in X.$ ■

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